Eur. Phys. J. C 55, 547–552 (2008)
DOI 10.1140/epjc/s10052-008-0620-8

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Neutrino mixing and the exponential form of the Pontecorvo–Maki–Nakagawa–Sakata matrix

G. Dattoli[1,a], K.V. Zhukovsky[2,b]

[1] Gruppo Fisica Teorica e Matematica Applicata Unita Tecnico Scientifica Tecnologie Fisiche Avanzante (FIS-MAT), ENEA – Centro Ricerche Frascati, Via Enrico Fermi 45, 00044 Frascati, Rome, Italy
[2] Dept. of Optics and Spectroscopy, Faculty of Physics, M.V. Lomonosov Moscow State University, Moscow 119991, Russia

Received: 19 November 2007 / Revised version: 25 March 2008 /
Published online: 27 May 2008 –

Abstract. The form of the neutrino mixing matrix is discussed. The exponential parameterisation of the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix is proposed and the generation of the new unitary parameterisation of the neutrino mixing matrix by the exponential form is demonstrated. The CP violating phase and the Majorana phases in the PMNS matrix are accounted for by a special term, separated from the rotational one. The O(3) rotation matrix in the angle-axis form is discussed in the context of such a representation of the mixing matrix. Its properties are reviewed in the context of the unitarity requirement.

1 Introduction

One of the major achievements in physics of the 20th century is the formulation of the so called Standard Model (SM) [1–3], which gives a joint description of the electromagnetic and weak interactions by a single theory. The SM has been extensively tested during the last decades. When the existence of the neutral weak interactions was confirmed and the intermediate vector bosons (W^-, W^+ and Z^0) were discovered with properties exactly as predicted, the SM really proved reliable.

The Standard Model represents one of the most beautiful and well structured models in physics. The description of the electroweak interactions in the framework of the standard model is implemented via gauge theory, based on the $\mathrm{SU}(2)\otimes\mathrm{U}(1)$ group, and its spontaneous breaking is accounted for via the Higgs mechanism. The matter fields are organised in families, where the left handed fermions are coupled into weak isodoublets and the right handed components transform as weak isosinglets.

The gauge bosons play the role of the interaction carriers in the theory – they are the W^- and the W^+ charged weak current bosons, the Z^0 neutral weak current boson, and the photon, the QED interaction carrier.

There is a surprising analogy between the structures of the lepton pairs:

$$\varphi_e=\begin{pmatrix}\nu_e\\ e\end{pmatrix}_{\mathrm{L}},\quad \varphi_\mu=\begin{pmatrix}\nu_\mu\\ \mu\end{pmatrix}_{\mathrm{L}},\quad \varphi_\tau=\begin{pmatrix}\nu_\tau\\ \tau\end{pmatrix}_{\mathrm{L}},\qquad(1)$$

and the left components of the quark spinors, composing weak isospin pairs:

$$\begin{pmatrix}u\\ d'\end{pmatrix}_{\mathrm{L}},\quad \begin{pmatrix}c\\ s'\end{pmatrix}_{\mathrm{L}},\quad \begin{pmatrix}t\\ b'\end{pmatrix}_{\mathrm{L}}.\qquad(2)$$

Neutrino physics has an important place in the standard model. The discovery of the neutrino masses and evidence that neutrinos actually do change flavour in nature occurred in experiments on the mixing of solar (see, for example, [4, 5]), atmospheric [6] and reactor neutrino [7], contrary to what was originally supposed. The situation with neutrino mixing is somewhat more complicated than with quark mixing and our understanding of this phenomenon is vaguer. Neutrino mixing was originally invented by Pontecorvo [8–10]. The standard model can be adapted to account for massive neutrinos, and the neutrinos mass term can be incorporated in the framework of the SM seamlessly via a procedure similar to the one that leads to the appearance of the quark mass terms. However, the neutrinos may have another source of mass, which comes through the Majorana mass term.

Flavour mixing in the lepton sector can be formalised in a way similar to the well known mixing in the quark sector. Massive neutrinos mean the existence of a spectrum of at least three neutrino mass eigenstates and mixing of the neutrino mass states $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ or $\boldsymbol{\nu}_3$ occurs. The state $\boldsymbol{\nu}_e$, $\boldsymbol{\nu}_\mu$ or $\boldsymbol{\nu}_\tau$, built of linear combinations of neutrino states with different masses, analogous to that of the bottom components of the quark pairs in (2), can be written as follows:

$$|\nu_\alpha\rangle=\sum_{i=1,2,3}U^*_{\mathrm{PMNS}_{\alpha i}}|\nu_i\rangle\,,\quad U_{\mathrm{PMNS}_{\alpha i}}\equiv\langle\nu_\alpha|\nu_i\rangle\,,\qquad(3)$$

[a] e-mail: dattoli@frascati.enea.it
[b] e-mail: zhukovsk@casaccia.enea.it, zhukovsk@phys.msu.ru

where U_{PMNS} is the unitary leptonic mixing matrix, or Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix [11]. Generally speaking, lepton mixing means that the charged W boson can couple to any charged lepton mass eigenstate $(e, \boldsymbol{\mu}, \boldsymbol{\tau})$ and any neutrino mass eigenstate. For example, the $U_{\alpha i}$ would be the amplitude of the decay of the W^+ boson into a pair of a lepton of type α and a neutrino of type i. Then the associative production of the lepton of such type α and the neutrino state of the type α implies the superposition of all neutrino mass eigenstates. It should be mentioned that, in fact, there are some indications [12–14] of more than three neutrino mass eigenstates:

$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}|\nu_i\rangle\,, \tag{4}$$

where $i = 1, 2, 3, 4$. The fourth neutrino does not couple to the standard model W and Z bosons and therefore it is called a sterile neutrino. However, in the present article we will assume just three generation mixing and consider only the 3×3 unitary neutrino mixing matrix. The matter effects, such as neutrino scattering on particles that they meet on the way through, for example, earth or sun can modify their propagation, but we will leave these questions [15–17] beyond the range of the present work.

2 Mixing matrix parameterisation proposals

The mixing matrix U can be written in several forms. When only two mass eigenstates and two correspondent flavour eigenstates are a good approximation, the 2×2 matrix is used:

$$U' = \begin{matrix} & \nu_1 & \nu_2 \\ \nu_\alpha & \cos\theta & \sin\theta \\ \nu_\beta & -\sin\theta & \cos\theta \end{matrix}\,. \tag{5}$$

When the standard three neutrinos theory is considered, the 3×3 mixing matrix is used. Unless the experiment shows that the 3×3 matrix is not unitary and requires the sterile neutrino or whatever else, the three generation mixing is considered with the following popular parameterisation of the mixing matrix U [18]:

$$U_{\mathrm{PMNS}} = U P_{\mathrm{Mjr}}\,, \tag{6}$$

where

$$P_{\mathrm{Mjr}} = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\alpha_1/2}, \mathrm{e}^{\mathrm{i}\alpha_2/2}, 1) \tag{7}$$

and

$$U = \begin{matrix} & \nu_1 & \nu_2 & \nu_3 \\ \nu_\alpha & c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ \nu_\mu & -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & s_{23}c_{13} \\ \nu_\tau & s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{23}c_{13} \end{matrix}\,, \tag{8}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, and the indices assume values $i, j = 1, 2, 3$. The mixing is given by the three mixing angles θ_{12}, θ_{23} and θ_{13}, and the CP violating phases δ, α_i and α_2. The phases α_1 and α_2 are non-zero only if the neutrinos are Majorana particles, which effectively means that they are identical to their antiparticles; then these phases have physical consequences only in lepton number violating processes. The corresponding phases α_1 and α_2 do not influence the neutrino oscillations regardless of whether neutrinos are Majorana particles or not.

Apart from Majorana phases, the term U in the parameterisation (8) is identical to the well known CKM matrix for quark mixing [18, 20–23]. The mixing angles θ_{12} and θ_{23} have quite well been determined experimentally as follows [18, 19, 24–26]:

$$\theta_{12} \cong 33.9 \pm 2.4^\circ\,, \tag{9}$$

$$\theta_{23} \cong 45 \pm 7^\circ\,. \tag{10}$$

It appears that these two mixing angles, θ_{12} and θ_{23}, are large and the third one, θ_{13}, is relatively small; and at the same time, many experimental analyses cannot account for very small angles, which do not impose a strict experimental bound on the values of θ_{13} [19, 27], setting the following approximate range:

$$\theta_{13} \le 13^\circ\,. \tag{11}$$

This fact is in striking contrast with the CKM matrix, where all three angles are small and hence approximate parameterisations (see, for example, [28]) can easily be constructed, based on an expansion into a power series of the parameter $\lambda = \sin\theta_{\mathrm{Cabibbo}} \approx 0.22$.

Other parameterisations of the neutrino mixing matrix can be found, for example, in [29–34], of which the tri-bimaximal mixing (TBM) form [35] of U_{PMNS} is most important. It is distinguished for the consistence with current experimental data. So far no convincing reasons for it to be exact exist and the approximate parameterisations of the PMNS matrix are built, based on deviations from the TBM form [35, 36]. Different from the parameterisations in the quark sector, which can be constructed with a single parameter, such approximate parameterisations for the neutrino mixing include three parameters, defining the deviations of the reactor, solar and atmospheric neutrino mixing angles from their tri-bimaximal values.

However, we can build a new exactly unitary parameterisation for the neutrino mixing, identical, except for the Majorana phases, to the exponential parameterisation of the CKM matrix [37, 38]:

$$U = \exp A\,, \tag{12}$$

where the argument of the exponent can be written as follows:

$$A = \begin{pmatrix} 0 & \lambda_1 & \lambda_3\mathrm{e}^{\mathrm{i}\delta} \\ -\lambda_1 & 0 & -\lambda_2 \\ -\lambda_3\mathrm{e}^{-\mathrm{i}\delta} & \lambda_2 & 0 \end{pmatrix}\,. \tag{13}$$

Since the angles θ_{12} and θ_{23} are of the same order of 1, we cannot establish the same hierarchy, as was done with the single parameter λ in the correspondent matrix in the quark sector, setting $\lambda_3 \propto \lambda^3$ and $\lambda_2 \propto \lambda^2$. The anti-Hermitian form of the matrix A ensures the unitarity of the mixing matrix U; see (12) [39].

The parameter δ accounts for the CP violation, and the parameters λ_i are responsible for the neutrino flavour mixing. Note that with $\delta = 0$, (13) turns into the three dimensional rotation matrix in angle-axis representation [38].

3 Exponential mixing matrix and the three dimensional rotations

Now, employing the exponential parameterisation (12) and (13), we can distinguish the separate rotational term and represent the mixing matrix in the form of the product, including the CP violating part, thus generating the following new unitary parameterisation, which can be considered for the neutrino mixing:

$$\tilde{V} = P_{\mathrm{Rot}} P_{CP} P_{\mathrm{Mjr}} \,. \tag{14}$$

In what follows we will show that the new parameterisation (14) is unitary and satisfies the necessary requirements for the mixing matrix.

The rotational part in (14) is given by

$$P_{\mathrm{Rot}} = \mathrm{e}^{A_{\mathrm{Rot}}} = \exp\begin{pmatrix} 0 & \lambda & \mu \\ -\lambda & 0 & -\nu \\ -\mu & \nu & 0 \end{pmatrix}, \tag{15}$$

and the CP violating part consists of the matrix

$$P_{CP} = \mathrm{e}^{A_{CP}} \,, \tag{16}$$

with

$$A_{CP} = \begin{pmatrix} 0 & 0 & \mu\left(-1+\mathrm{e}^{\mathrm{i}\delta}\right) \\ 0 & 0 & 0 \\ \mu\left(1-\mathrm{e}^{-\mathrm{i}\delta}\right) & 0 & 0 \end{pmatrix} \tag{17}$$

and the Majorana part can be written in the form of the following exponent:

$$P_{\mathrm{Mjr}} = \mathrm{e}^{A_{\mathrm{Mjr}}} \,, \tag{18}$$

where the argument of the exponent is given by the Majorana matrix

$$A_{\mathrm{Mjr}} = \mathrm{i}\begin{pmatrix} \alpha_1/2 & 0 & 0 \\ 0 & \alpha_2/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

It should be said that the same approach, successfully applied for the quark mixing matrix in [38] with account for the smallness of the parameter λ in the quark mixing matrix, allowed for establishing the approximate equality of the new generated matrix and the CKM matrix. The neutrino mixing matrix does not contain such a small parameter and the values of the phase δ as well of the phases α_1 and α_2 so far remain unconstrained by experiments. The rotational part of the parameterisation (14), written in the form (15), is nothing else but the angle-axis form of a three dimensional rotation M:

$$M = \begin{pmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{pmatrix}, \tag{20}$$

defined by a single angle of rotation Φ and a direction unit vector $\hat{n} = (n_x, n_y, n_z)$ of the fixed axis, around which the rotation is performed:

$$M(\hat{n}, \Phi) = \mathrm{e}^{\Phi N}\,, \quad N = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}, \quad \hat{n} = (n_x, n_y, n_z)\,. \tag{21}$$

Generally speaking, the rotation angles in the matrix (14) do not necessarily coincide with those constituting the elements $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ of the PMNS matrix in the standard parameterisation (8). It occurs only when the CP violating phase is small, because we can no more count on the small value of the parameter λ for the neutrino mixing as was the case for quark mixing, and the phases δ, α_1 and α_2 are still experimentally undetermined.

Let us denote the diagonal components of the matrix (20) as M_i and, thus, omitting the Majorana part, the parameterisation (14) can be written as follows:

$$\tilde{U} = M P_{CP} = \begin{pmatrix} M_x \cos 2\Delta + \kappa^- M_{xz} \sin 2\Delta & M_{xy} & M_{xz}\cos 2\Delta + \kappa^+ M_x \sin 2\Delta \\ M_{yx}\cos 2\Delta + \kappa^- M_{yz} \sin 2\Delta & M_y & M_{yz}\cos 2\Delta + \kappa^+ M_{yx}\sin 2\Delta \\ M_{zx}\cos 2\Delta + \kappa^- M_z \sin 2\Delta & M_{zy} & M_z \cos 2\Delta + \kappa^+ M_{zx}\sin 2\Delta \end{pmatrix}, \tag{22}$$

or

$$\tilde{U} = \left(\Xi_1 \cos 2\Delta + \kappa^- \Xi_3 \sin 2\Delta,\, \Xi_2,\, \Xi_3 \cos 2\Delta + \kappa^+ \Xi_1 \sin 2\Delta\right), \tag{23}$$

with the following vector parameters:

$$\Xi_1 = \begin{pmatrix} M_x \\ M_{yx} \\ M_{zx} \end{pmatrix}, \quad \Xi_2 = \begin{pmatrix} M_{xy} \\ M_y \\ M_{zy} \end{pmatrix}, \quad \Xi_3 = \begin{pmatrix} M_{xz} \\ M_{yz} \\ M_z \end{pmatrix}, \tag{24}$$

where

$$\Delta = \mu \sin\frac{\delta}{2}\,, \quad \kappa^{\pm} = \mathrm{i}\mathrm{e}^{\pm\mathrm{i}\frac{\delta}{2}} \tag{25}$$

and the rotation matrix is determined by the following well known tensor form (see, for example, [40]):

$$M_{ij} = (1-\cos\Phi) n_i n_j + \delta_{ij}\cos\Phi - \varepsilon_{ijk} n_k \sin\Phi\,, \tag{26}$$

where δ_{ij} is the Kronecker symbol, ε_{ijk} is the Levi-Civita symbol, n_i are the components of the rotation vector and Φ is the rotation angle. From a comparison of the matrices (22) with account for (25), (26) and (8) one can express the angle of the rotation Φ and the rotation vector n in terms of the parameters of the standard form of the PMNS mixing matrix c_{ij} and s_{ij}. However, these relations appear cumbersome and we omit them here. Comparing the rotational generator in three dimensions (21) with the rotational part (15) of the exponential parameterisation of the PMNS matrix, we obtain the coordinates of the rotation unit vector $\hat{n}$ in terms of the parameters of the exponential parameterisation of the PMNS matrix (14) as follows:

$$n_x = \frac{\nu}{\Phi}\,, \quad n_y = \frac{\mu}{\Phi}\,, \quad n_z = -\frac{\lambda}{\Phi}\,, \tag{27}$$

where the rotation angle Φ can be expressed in terms of the parameters of the exponential mixing matrix as follows:

$$\Phi = \pm\sqrt{\lambda^2+\mu^2+\nu^2}\,. \tag{28}$$

The simple formulae (27) and (28) relate the parameters of the purely rotational part of the neutrino mixing matrix in exponential parameterisation to the angle and the axis of the three dimensional rotation. From a comparison of the angle-axis form (21) with the TBM form of the mixing matrix [35], which is in good agreement with the experimental data, and accounting for (27) and (28), we obtain the following values for the parameters of the exponential parameterisation (15):

$$\lambda \cong 0.5831\,, \quad \mu \cong -0.2415\,, \quad \nu \cong -0.7599\,. \tag{29}$$

Note that the absolute value of the parameter μ is rather small, compared with 1, and hence we could possibly make use of it, performing the series expansion in this small parameter in certain cases. Then the matrix (8) can be written as follows:

$$U \cong P_{\mathrm{Rot}} P_{CP}\,, \quad \mu\delta \ll 1\,. \tag{30}$$

In this approximation $P_{\mathrm{Rot}} = M$ (see (20)). However, we will not employ such an expansion in this work because the condition in the above equation is fulfilled only roughly, contrary to the case of quarks, where it is satisfied very well. Moreover, calculating the coordinates of the rotation vector (27) for the exponential form of the neutrino mixing matrix and the quark mixing matrix from the experimental data (see, for example, [18]) we surprisingly find the value of the angle between these vectors to be approximately 44.5°. The fact that the rotation axes for the neutrino and the quark mixing form an angle very close to 45° is another way to formulate the hypothesis of complementarity and equality for the mixing angles of quarks and neutrinos [41, 42]. We hope to address this question in forthcoming publications.

4 Exponential parameterisation of the CP violating terms in the mixing matrix

In the previous section we have distinguished the real and the imaginary parts in the exponential parameterisation, thus separating the rotational term (15) in the PMNS matrix from the CP contribution. Now we can combine the CP violating term and the Majorana term into one, including the parameter δ and α, with the help of the following identity:

$$\mathrm{e}^{A_{CP}}\mathrm{e}^{A_{\mathrm{Mjr}}} \cong \mathrm{e}^{A_{CP}+A_{\mathrm{Mjr}}}\left(1+\frac{1}{2}[A_{CP}, A_{\mathrm{Mjr}}]\right), \tag{31}$$

where the commutator is of the order of $O(\alpha_1\mu\delta)$. Whether the neutrinos are Majorana particles or not is still unknown and in any case α does not enter the oscillation phenomena, whereas the phase δ is nonzero only if the CP symmetry is not conserved, which is expected but still not confirmed experimentally. On the supposition that the parameters α and δ are small, the product of (16) and (18) takes the following approximate form:

$$V_{\mathrm{MCP}} = P_{CP}P_{\mathrm{Mjr}} \cong \exp\begin{pmatrix} \mathrm{i}\frac{\alpha_1}{2} & 0 & (-1+\mathrm{e}^{\mathrm{i}\delta})\mu \\ 0 & \mathrm{i}\frac{\alpha_2}{2} & 0 \\ (1-\mathrm{e}^{-\mathrm{i}\delta})\mu & 0 & 0 \end{pmatrix} + O(\alpha_1\mu\delta)\,, \tag{32}$$

being itself a unitary matrix. Moreover, avoiding any assumption of the smallness of the phases, the direct computation of the product of $P_{CP}P_{\mathrm{Mjr}}$ yields the following result:

$$V_{\mathrm{MCP}} = P_{CP}P_{\mathrm{Mjr}} = \begin{pmatrix} \xi_1\cos 2\Delta & 0 & \kappa^+\sin 2\Delta \\ 0 & \xi_2 & 0 \\ \xi_1\kappa^-\sin 2\Delta & 0 & \cos 2\Delta \end{pmatrix}, \tag{33}$$

where

$$\xi_{1,2} = \mathrm{e}^{\mathrm{i}\frac{\alpha_{1,2}}{2}}\,. \tag{34}$$

The V_{MCP} matrix – the factor in the exponential parameterisation of the PMNS matrix, responsible for the CP violation effects – can be expressed in terms of the Bessel functions, taking into account the series expansion of the exponent:

$$\exp[\mathrm{i}x\sin\alpha] = \sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n\alpha}J_n(x)\,. \tag{35}$$

Hence, the V_{MCP} matrix takes the following form:

$$V_{\mathrm{MCP}} = \begin{pmatrix} \xi_1\sum\limits_{n=-\infty}^{\infty}J_n(2\mu)\cos\left(\frac{n\delta}{2}\right) & 0 & \kappa^+\sum\limits_{n=-\infty}^{\infty}J_n(2\mu)\sin\left(\frac{n\delta}{2}\right) \\ 0 & \xi_2 & 0 \\ \xi_1\kappa^-\sum\limits_{n=-\infty}^{\infty}J_n(2\mu)\sin\left(\frac{n\delta}{2}\right) & 0 & \sum\limits_{n=-\infty}^{\infty}J_n(2\mu)\cos\left(\frac{n\delta}{2}\right) \end{pmatrix}, \tag{36}$$

where the CP non-conserving parameter δ is separated from the parameter μ in the arguments of the Bessel functions of the first kind $J_n(2\mu)$. Note that this result is exact in terms of δ and α_1, α_2 in the sense that we did not assume that they were small.

Note that the matrix V_{MCP} is not symmetric with respect to the parameters $\xi_{1,2}$ and the Majorana term can interplay with the δ phase, in particular, in the (3,1) entry. When $\alpha_1 = 0$, the symmetric form of the matrix (33) is restored. For $\xi_2 = 1$, i.e. $\alpha_2 = 0$, the form of the V_{MCP} matrix reminds one of the forms of the mixing matrix for just two lepton generations, acting on an electron, tauon and correspondent neutrinos with the weights ξ_1 for the (1,1) entry of the mixing matrix, κ^+ for the (3,1) entry and $\xi_1\kappa^-$ for the (1,3) entry of the mixing matrix. For a non-zero value of α_2 the vector of the mixed neutrino states under the action of the V^*_{MCP} matrix can be written as follows:

$$|\nu_\alpha\rangle = \begin{pmatrix} |\nu_1\rangle\xi_1^*\cos 2\Delta + |\nu_3\rangle\kappa^{+^*}\sin 2\Delta \\ |\nu_2\rangle\xi_2^* \\ |\nu_1\rangle\xi_1^*\kappa^{-^*}\sin 2\Delta + |\nu_3\rangle\cos 2\Delta \end{pmatrix}. \qquad (37)$$

A direct check confirms the existence of the Hermitian conjugated matrix, inverse with respect to V_{MCP}, which ensures the unitarity of V_{MCP}:

$$V^{-1}_{\mathrm{MCP}} \cdot V_{\mathrm{MCP}} = V^{+}_{\mathrm{MCP}} \cdot V_{\mathrm{MCP}} = I \qquad (38)$$

and the unitarity of the exponential parameterisation (14) of the PMNS matrix:

$$\tilde{V}^{-1} \cdot \tilde{V} = \tilde{V}^{+} \cdot \tilde{V} = I\,. \qquad (39)$$

The action of the rotation matrix on the neutrino vector (37) yields the following final result for the neutrino vector after the PMNS transformation:

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = M \cdot V^*_{\mathrm{MCP}} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix} = F|\nu_1\rangle + G|\nu_2\rangle + H|\nu_3\rangle\,, \qquad (40)$$

where α denotes the type of the neutrino, M is for the rotation matrix in angle-axis representation, and

$$F = \xi_1^*\left(\Xi_1\cos 2\Delta + \Xi_3\kappa^{-^*}\sin 2\Delta\right), \qquad (41)$$

$$G = \xi_2^*\Xi_2\,, \qquad (42)$$

$$H = \xi_2^*\left(\Xi_1\kappa^{+^*}\sin 2\Delta + \Xi_3\cos 2\Delta\right). \qquad (43)$$

The parameters Δ, κ and ξ in (41)–(43) are given by (25) and (34), the vector parameters Ξ_i are given by (24), where M_{ij} are the entries (26) of the rotation matrix M and the components of the rotation vector n, and the rotation angle Φ can be expressed in terms of the parameters of the mixing matrix μ, λ, ν as demonstrated in (27) and (28). Thus, the contribution due to ν_2 neutrinos is affected by the α_2 Majorana phase (42), whereas the ν_1 and ν_3 neutrinos enter respectively with α_1 and α_2 dependent factors (41) and (43). Furthermore, (41)–(43) can be seen as a sort of rotation in the angle 2Δ, determined by the CP phase δ (25) with the weights Ξ and κ and their products.

5 Conclusions

Mixing in the lepton and quark sectors have much in common and its study may lead to a new understanding of the standard model, the term “flavour” and its role and the reason why there are three (or more?) lepton and quark generations. In the present work we have drawn the parallel between the CKM matrix in the standard model and the PMNS matrix for neutrino mixing. The geometric nature of the PMNS matrix without CP violations and Majorana effects reduces the neutrino mixing to the well known mechanical problem of a Rodriguez rotation. Indeed, in the case of conserved CP, the neutrino mixing in the SM can be viewed as a rotation around a fixed axis in 3D space by the angle Φ. For $\Phi = 0$ the mixing between the neutrinos fades out, since the mixing matrix – the rotation matrix – becomes the unit matrix I. The CP violating phases δ and α_i break this symmetry and hence the simple geometric picture based on the O(3) generators algebra does not hold any more. From the comparison with the experimental data and the tri-bimaximal form of the mixing matrix we carry out the following value for the angle $\Phi \cong 56°$ and conclude that the rotation axes for the neutrino and the quark mixing form an angle of $\approx 45°$, which is another way to formulate the hypothesis of complementarity and equality for the mixing angles of quarks and neutrinos [41, 42].

The exponential parameterisation of the PMNS matrix allows for the generation of new unitary mixing matrix parameterisation with a separated CP violating part. This representation of the mixing matrix may prove to be convenient to account for the matter–antimatter asymmetry in the universe, the evolution of which, perhaps, was influenced not only by a quark CP violating complex phase, but also by the leptonic CP violation [43] and lepton asymmetry [44, 45]. The CP violation due to the phase δ can be combined with the effect of the Majorana phases α_i and their interplay can be studied in the exponential form (32) and by (33). The CP violating terms due to the phase δ and the Majorana phases $\alpha_{1,2}$ are effectively separated in the coefficients of the series expansion of the Bessel functions in (36). The exact analysis of the problem of neutrino mixing in the presence of CP violating terms with the help of the mixing matrix in the form (6)–(8) becomes complicated and leads to a scarcely transparent physical picture. On the other hand, the exponential parameterisation of the PMNS mixing matrix transforms the neutrino vector, distinguishing the contribution of the CP violating phase for each neutrino type as demonstrated in formulae (40)–(43). We believe it can be appropriate for the analysis of experimental data in many neutrino oscillation experiments conducted at present and even more numerous experiments planned for future.

References

1. S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967)
2. A. Salam, In: Elementary Particle Theory, ed. by N. Svartholm (Almquist Forlag AB, Stockholm, 1968)
3. S.L. Glashow, Nucl. Phys. **22**, 579 (1961)
4. Super-Kamiokande Collaboration, M.B. Smy et al., Phys. Rev. D **69**, 011 104 (2004)

5. KamLAND Collaboration, T. Ataki et al., Phys. Rev. Lett. **94**, 081 801 (2005)
6. Super-Kamiokande Collaboration, Y. Ashie et al., Phys. Rev. D **71**, 112 005 (2005)
7. CHOOZ Collaboration, M. Apollonio et al., Eur. Phys. J. C **27**, 331 (2003)
8. B. Pontecorvo, Zh. Eksp. Teor. Fiz. **33**, 549 (1957)
9. B. Pontecorvo, Zh. Eksp. Teor. Fiz. **53**, 1717 (1967)
10. B. Pontecorvo, Sov. Phys. JETP **26**, 984 (1968)
11. Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. **28**, 870 (1962)
12. H. Murayama, T. Yanagida, Phys. Lett. B **520**, 263 (2001)
13. G. Barenboim et al., JHEP **0210**, 001 (2002)
14. M.C. Gonzalez-Garcia, M. Maltoni, T. Schwetz, Phys. Rev. D **68**, 053 007 (2003)
15. L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978)
16. S.P. Mikheev, A.Y. Smirnov, Yad. Fiz. **42**, 1441 (1985)
17. S.P. Mikheev, A.Y. Smirnov, Sov. J. Nucl. Phys. **42**, 913 (1985)
18. Particle Data Group, W.-M. Yao et al., J. Phys. G: Nucl. Part. Phys. **33**, 1 (2006)
19. H. Minako, K. Yee, O. Naotoshi, T. Tatsu, YITP **05-52**, 73P (2006) [arXiv:hep-ph/0602115]
20. L.-L. Chau, W.-Y. Keung, Phys. Rev. Lett. **53**, 1802 (1984)
21. H. Harari, M. Leurer, Phys. Lett. B **181**, 123 (1986)
22. H. Fritzsch, J. Plankl, Phys. Rev. D **35**, 1732 (1987)
23. F.J. Botella, L.-L. Chao, Phys. Lett. B **168**, 97 (1986)
24. SNO Collaboration, B. Aharmin et al., nucl-ex/0502021
25. KamLAND Collaboration, T. Araki et al., Phys. Rev. Lett. **94**, 081 801 (2005)
26. Y. Suzuki, presented at the XXII Int. Symp. on Lepton and Photon Interactions at High Energies (Lepton–Photon 2005), Uppsala, Sweden, July 2005
27. Chooz Collaboration, C. Bemporad, Nucl. Phys. Proc. Suppl. **77**, 159 (1999)
28. L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983)
29. Z.Z. Xing, Phys. Lett. B **533**, 85 (2002)
30. J.D. Bjorken, P.F. Harrison, W.G. Scott, Phys. Rev. D **74**, 073 012 (2006)
31. A. Datta, L. Everett, P. Ramond, Phys. Lett. B **620**, 42 (2005)
32. S. Antush, S.F. King, Phys. Lett. B **631**, 42 (2005)
33. G. Altarelli, F. Feruglio, Y. Lin, Nucl. Phys. B **775**, 31 (2007)
34. E. Ma, Mod. Phys. Lett. A **22**, 101 (2007)
35. S.F. King, arXiv:0710.0530 [hep-ph]
36. N. Li, B.Q. Ma, Phys. Rev. D **71**, 017 302 (2005)
37. G. Dattoli, K. Zhukovsky, Eur. Phys. J. C **50**, 817 (2007)
38. G. Dattoli, K. Zhukovsky, Eur. Phys. J. C **52**, 591 (2007)
39. G. Dattoli, E. Sabia, A. Torre, Nuovo Cim. A **109**, 1425 (1996)
40. H. Goldstein, Classical Mechanics (Addison-Wesley, Cambridge, MA, 1950), p. 399
41. H. Minakata, A.Y. Smirnov, Phys. Rev. D **70**, 073 009 (2004)
42. M. Raidal, Phys. Rev. Lett. **93**, 161 801 (2004)
43. M. Fukugita, T. Yanagida, Phys. Lett. B **174**, 45 (1986)
44. V. Kusmin, V. Rubakov, M. Shaposnikov, Phys. Lett. B **155**, 36 (1985)
45. S. Pascoli, S. Petcov, W. Rodejohann, Phys. Rev. D **68**, 093 007 (2003)